Notations Across Cultures for Teaching

Paul Libbrecht
CERMAT
University of Education
Karlsruhe, Germany
http://www.hoplahup.net/paul/

Maria Droujkova
Natural Math LLC,
Cary,
North Carolina, USA
http://www.naturalmath.com/

Erica Melis[1]
German Center for Research
in Artificial Intelligence,
Saarbrücken, Germany
http://www.dfki.de/~melis/

Abstract *This paper reports about the census of mathematical notations, an initiative to collaboratively observe the diverging mathematical notations across cultures and languages. The census is the starting point for a type of learning activity for which multiple scientific evidence is available: the activity of inventing mathematical notations so as to stimulate the representation competencies. We explore the ingredients of such a learning activity and how the census helps stimulating it; we detail extensions of the census so as to serve the invented notations activities.*

Introduction

In this paper we propose to extend an existing college-level census of mathematical notations. We plan to extend it to more languages; to notations used in primary and secondary mathematics; to historical notations; and to student-invented notations.

Many persons we have met usually consider mathematics a *universal language*. This belief is directly expressed in software design: only words, but not formulæ or other representations are translated for localizations. However, mathematical notations differ considerably from one culture to another. For example, on the right are notations for half-open intervals in English, French, or Dutch.[2]

$$[a, b)$$

$$[0, \frac{\pi}{2}[$$

$$[0, \infty\rangle$$

Only a few publications are devoted to documenting this diversity along the cultures, such as the notable book of Florian Cajori (1928). If a part of mathematical media is not specifically focusing on history or diversity, it uses only one version of notation.

The Math-Bridge project launched a census of the mathematical notations. The goal was to raise the quality of software localizations, minimizing confusion in the subject which can become highly problematic for learners.[3] As an example of confusion

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[1] Erica Melis was instrumental in leading the works on ActiveMath. She passed away in February 2011, but her ideas and encouragement were important for the project. She is our coauthor in spirit.

[2] See more details about these notations at http://wiki.math-bridge.org/display/ntns/interval_co .

[3] Such confusion could be obtained, for example, in the English Wikipedia: in April 2009, two different notations for the binomial coefficient were used depending on whether it was on the Binomial coefficient or the Combination page.

consider MathML 2 (Carlisle et al 2001, section 5.4.1). It claims the binomial coefficient in Russian has the greater number at the top, and in French at the bottom.

In fact, this claim is the product of scholars copying one another: most Russian sources we have later consulted confirm that the binomial coefficient is written with the greater number at the bottom, similarly to the French notation. The Math-Bridge census we describe below allowed users to report examples of mathematical notations in widely used mathematical texts and thus prevent such confusion.

But is it necessary to always use **previously published** mathematical notations?

One of the main endeavors of mathematicians, scientists, and engineers is to invent or to modify notations, making discourse more efficient and readable. To prepare students for this part of doing real mathematics, several educational theories we review in Section 3 invite them to analyze multiple existing notations, or to create their own. *"Every student's educational activities should include the rich variety of experience and learning made possible through participation in multiple practices of representation"* (Greeno and Hall, 1997).

Despite the need recognized in theory, the pedagogical practices supporting learner-invented notations are rare. Such support must include developing teachers' pedagogical content knowledge (Mishra and Koehler, 2006) about multiple representations specific to particular topics, such as ways to guide students using the various representations relevant to the topics. It can only be developed by experiencing rich, topic-specific collections of representation examples. A teacher already devoted to supporting student representations gradually develops these collections, and with time is able to help students better. The census we plan to build will aggregate pedagogical content knowledge from multiple individuals and organize it by topic. It will powerfully support teacher educators and novice teachers and, we hope, provide a sufficiently deep awareness of the value of the notations in their context of use as was highlighted by Barwell and Kaiser (2005).

Moreover, making it available to the general public, including students, will dispel misconceptions about mathematical notations, such as the belief that they are already fully invented and standardized worldwide.

Outline

In Section 1 of the paper we first introduce the notation census implemented within the Math-Bridge project. Then we analyze the impact and reasons behind several topic-specific example spaces of notations from the census in Section 2. Section 3 reviews existing pedagogical theories supporting creation of notations. Section 4 distills a didactic recipe from these theories. Section 5 describes details of its implementation.

1. The census

The Math-Bridge notation census has been created for the Math-Bridge project which aims at using ActiveMath for entry-level, remedial university courses for five European countries. ActiveMath is a personalized learning environment (Melis et al., 2005) for mathematics, it exploits mathematical formulæ which are encoded in a semantic format, while still aiming at notations familiar to different learners. ActiveMath intends to support learners that migrate from one language environment to the other by offering the same content in multiple languages in parallel so that users can migrate from language to the other in the middle of learning. The census was created originally as a method to gather the knowledge to construct the rendering tools for ActiveMath

The census is available at http://wiki.math-bridge.org/display/ntns/Home . It is currently based on the Confluence wiki platform of Atlassian. It is a **verifiable** and **open** repository of notations in wide use, with the following features:

- **Sources** are collected within the bibliography. They are mathematical textbooks widely used in a context, such as a country or a branch of science.
- The name of a submitter is associated with each source for **traceability**.
- Notations are grouped by web pages **tagged by meaning**, also called semantics. We rely on the OpenMath content-dictionaries repository (Davenport et al., 2009) to provide the basic names of the elements of semantics.
- The census consists of **observed and pictured examples of use**. Each example is a picture (screenshot or scan) of the usage in a references source.

As of writing this, the census contains notations for much of elementary calculus.

2. Examples showing the need to customize notations

> *"Although we may assume that mathematics is easier than other subjects for students who are English learners, through the belief that the language of numbers is universal, the reality is that there are many differences in notation, conventions, and algorithms. Knowing more about the diverse algorithms students bring to the classroom and their ways of recording symbols for "doing mathematics" will assist you in supporting students and responding to families, particularly knowing that what we may call a "traditional algorithm" is not the tradition in other countries. Awareness of alternative algorithms will help you explore the procedures and ways to record answers that your students know from prior experiences in schools outside of the United States or from approaches taught to them by their families."* (van de Walle et al 2010, addressing US teachers, p. 217)

Math-Bridge's census captures several dimensions of variability among notations. An easily observed dimension is the variability by **language**: for example, here are the greatest common divisor notations in wide use in Spanish, German, Arabic, English, Dutch, and Finnish:[4]

mcd ggT ق.م.أ gcd GGD syt

Here the least common multiple notations in German, Spanish, Finnish, and Arabic:

kgV $m.c.m.$ pym م.م.أ

These notations are abbreviations hence depend on the language.

New notations are made for newly invented concepts and those already in use to enhance the clarity within a given context. An example is the dependency on a scientific context: the square root of -1 is denoted by i in Western mathematics and by j in Western electrical engineering, to differentiate it from the symbol for intensity.[5]

In addition to differences by language, a given text frequently introduces unique notations, and scientists expect notations to be **declared** at the start. Historically, mathematicians and physicists have rich exchanges where informally introduced notations, sometimes initially considered notation abuses, lead to the development of new theories. For example, Laurent Schwartz introduced generalized functions in the 1940s to be able to represent such *functions* as the Dirac's delta function.

These examples of national and historic dimensions of sign interpretations show the importance of context for understanding notation. Most current curricula attempt to simplify mathematics for learners by removing this semiotic ambiguity altogether. This approach is impossible in the globalized world, and as we argue below, leads to missed teaching opportunities and to increased learning difficulties.

3. Theories calling for student-invented notations

In this section, we list educational theories where invented notation plays an important role, situating our project in relation to existing education research frameworks. Researchers who study educational semiotics (Ernest, 2008) necessarily invite students to interpret, and frequently invent, signs and symbols, which is seen as an integral part of learning. In particular, research on multiple representations in mathematics focuses on types of representations, and connections between types and particular mathematical topics (Davis and Maher, 1997).

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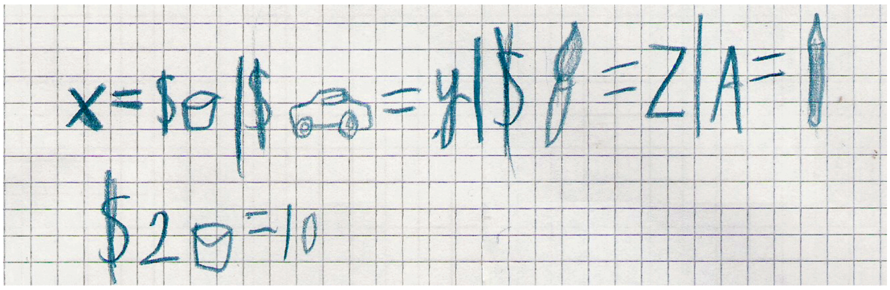
[4]See the census page about gcd: http://wiki.math-bridge.org/display/ntns/gcd or lcm http://wiki.math-bridge.org/display/ntns/lcm .

[5]For references for this affirmation, see http://wiki.math-bridge.org/display/ntns/nums1_i .

Social constructivists and enactivists focus on student-invented notations as a part of mathematical communication (Cobb et al., 2000). In these theories, students use representations, including invented ones, for creating shared understanding.

Constructionism (Kafai and Resnick, 1996) connects learning to making tangible or computer-based objects. An important facet of constructionism is creation of notations to assist the making, as well as communication about made objects.

The modeling approaches (Lesh and Doerr, 2003) focus on students creating mathematical models of events or objects. Because models are created in mathematical notations, the act of re-presenting is a key part of these theories. All modeling theories provide students with notations, and most invite students to design their own. In the example below, which requires contextual data our project will supply for understanding, a student modeling prices uses invented notation with existing symbols, numerals, letters and pictograms combined in a novel way.



Researchers studying metaphors of individual learners (English, 1997) note that metaphors vary widely from person to person. Invented notation and language is among the primary ways to access and then to influence learners' metaphors.

Humanistic mathematics (White, 1993) focuses on expression of mathematics through the languages of the arts, including student-created representations. Ethnomathematics (Ascher, 1994) mostly focuses on historic uses of mathematics within different cultures, which includes explicating authentic representations.

Studies of problem-solving and invented strategies (Van de Walle at al., 2010) include the attention to invented representations. Researchers underlie the complexity of teacher attitudes and skills required to understand, debug, and develop novel student strategies and notation systems, including the knowledge of a wide range of examples of student-invented representations.

4. The didactic recipe

We integrated compatible ideas from the educational theories listed above, as well as pedagogical practices associated with them, to create a didactic recipe for learning activities that involve notation. Here, we briefly describe the significance of each ingredient in the recipe. In the next section we describe the proposed implementation.

- In the area of beliefs and attitudes, students and teachers need to **view notations as human-made and changeable**. This is contrasted to viewing notations as *sacred commandments* that are created by nameless authorities and last forever. A varied collection of notations used in different places in the world, and the history of each notation's changes through time, with names of inventors and the works, provides the view of notation as a human endeavor.
- Teachers and curriculum designers need **an example space of widely accepted and historical notations** by topic. For example, function notations usually include formulæ, graphs and tables of values. Number notations may be positional or not, hieroglyphic or composite. Descriptions of accepted, peer-reviewed notations from the past, indexed by topic, provides a teacher with a rich background even before running an activity for the first time, and serves a source of reference for curriculum designers.
- A space for **niche and invented notations**, separated but organized similarly to the census of widely used notations, sends the message that creating notations is an integral part of mathematical practices at all levels. This message is particularly strengthened by notations created by students.
- An explicit **invitation for site visitors to submit observations of notations** further strengthens the message that everyone is involved. This invitation takes the form of social participation tools, such as display of recent submissions or display of submissions by user. Whether students submit observations themselves or through parents and teachers, they see their work as a part of the continuum that spans international and historical mathematical notations.

This recipe supports central and peripheral participation within the global community of practice that uses mathematical notations as its social objects (Roth, McGinn 1998). Every student works locally, with educators who are informed by the previous content of the census. Such local groups, in turn, contribute notations they develop, thus becoming a part of the long-term work of mathematicians and mathematics educators, as captured in peer-reviewed media or historical documents. Such an approach makes notations invented by each student or teacher visible through the global census that is searchable, linkable and remixable. Within the collection, all the notations, including historical, current mainstream, niche, even invented and briefly used by individuals, coexist in one shared semantic space.

5. Summary of project dimensions

To achieve the recipe above, we will build on the existing notation census, extending it along several dimensions. The first is the **historical dimension**: the census currently provides observations in mathematics textbooks in wide use. When possible, links are made to the first occurrence or the definition of the symbol in the source. Some of the sources provide a historical perspective about the notations, but this is rather rare. Aggregating the history of notations will help to understand them better; hyperlinking to the precisely relevant historical statement provides a stronger traceability.

The second extension is the collection of **invented notations**. Grouping the notations by meaning, as described in the census section, provides guidance to users who wish to view notations others created. Labelling by source will be replaced with labelling by learning context, a more general database field describing the learning activity where notation invention takes place. The page with observations of invented notations, inviting judgement-free creativity, will be clearly differentiated from pages collecting verifiable, reviewed sources. The pages will be linked to one another.

Contributions to the invented notations will be openly accepted from everyone, just as the existing census is an open wiki. To promote wider contributions, we will put together a platform that is easier to use for submission and browsing.

We anticipate much interest in the project not only from the practitioners, but also from researchers using mathematics education frameworks described in Section 3. For example, research interests of the authors include modeling, early algebra, and multiplicative reasoning. We and our research groups plan to use the platform for sharing results of our studies with the public, and as a data source for further studies. The census will become a central hub providing a unique view on the world of mathematics, from the perspective of mathematical notations, which are a central instrument in contemporary science and mathematics education.

Acknowledgements

This research has been partially funded by the European commission under the eContentPlus project Math-Bridge (ECP-2008-EDU-428046) and by the institutions of the authors.

Bibliography

• Ascher, M. (1994). Ethnomathematics: A multicultural view of mathematical ideas. Chapman & Hall/CRC.

• Barwell R. and Kaiser G., Introduction: Mathematics education in culturally diverse classrooms, ZDM Zentralblatt für Didaktik der Mathematik, Volume 37, Number 2 (April 2005), Springer Verlag. Available online at http://www.emis.de/journals/ZDM/zdm052i.html .

• Cajori, F. (2007). *A History of Mathematical Notations: Vol. I*. Cosimo Classics. Retrieved from http://www.archive.org/stream/historyofmathema031756mbp

• Carlisle, D., Ion, P., Miner, R., Poppelier, N. (Eds.). (2003). Mathematical Markup Language (MathML) version 2.0 . W3C recommendation. World Wide Web Consortium. Retrieved from http://www.w3.org/TR/MathML2/

• Cobb, P., Yackel, E., & McClain, K. (2000). Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design. Lawrence Erlbaum.

• Davenport, J. (Ed.). (2010). OpenMath. Retrieved from http://www.openmath.org/cd/

• Davis, R., & Maher, C. (1997). How students think: The role of representations. In L. English (Ed.), *Mathematical reasoning: Analogies, metaphors and images*, Studies in Mathematics Thinking and Learning (pp. 93-116). Mahwah, NJ: Lawrence Erlbaum.

• English, L. (Ed.). (1997). Mathematical reasoning: Analogies, metaphors, and images. Mahwah, NJ: Lawrence Erlbaum.

• Ernest, P. (2008), "Towards a semiotics of mathematical text," For the learning of mathematics: an international journal of mathematics education 28 (2008): 42.

• Greeno, J., & Hall, R. (1997). Practicing representation: Learning with and about representational forms. *Phi Delta Kappan*, *78*(5).

• Kafai, Y., & Resnick, M. (1996). Constructionism in Practice: Designing, Thinking, and Learning in A Digital World (1st ed.). Routledge.

• Lesh, R., & Doerr, H. (2003). Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching. Mahwah, N.J.: Lawrence Erlbaum.

• Melis, E., Goguadze, G., Homik, M., Libbrecht, P., Ullrich, C., & Winterstein, S. (2006). Semantic-aware components and services of ActiveMath. *British Journal of Educational Technology*, *37*(3), 405–423.

• Mishra, P., & Koehler, M. (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers College Record*, *108*(6), 1017–1054.

• Roth, W., & McGinn, M. (1998). Inscriptions: Toward a theory of representing as social practice. *Review of educational research*, *68*(1), 35.

• Van de Walle, J., Karp, K., & Bay-Williams, J. (2010). Elementary and middle school mathematics: Teaching developmentally (7th ed.). Pearson Education.

• White, A. (1993). *Essays in humanistic mathematics*. Mathematical Association of America.