From Concept to Teaching Resource

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Abstract. Through a historical overview, we show how a scientific innovation can become a part of knowledge which is mature enough to be taught: set theory. This example shows through explicit representations across history how much richer knowledge is needed in order to be taught and, thus, to be made into a tutoring system than its scientific basis. The historical process explains us how an idea needs to be adjusted, to be turned into practical exercises, notations, teaching units, and practical recommendations to include it into the curriculum standards. The varying representations are sketched, be them notational, graphical, haptic, textual, or even part of artists productions.
The paper sketches paths of how the cultures developed around a mathematical concept needed to evolve in order to become teachable.

Keywords: Mathematical Cultures · Learning Resources · Teaching Practice · Mathematical Concepts · Notations · Set Theory.

1 Motivation

Our society has evolved thanks to the evolution of knowledge powered by new concepts and ideas created by scientists. The need for science is becoming bigger every day with the science subjects themselves evolving rapidly (consider the growing appearance of "Data Science" which emerged in 2001 but is now becoming a needed competence in most engineering subjects). To support this evolution, the educational systems regularly must adjust their curriculum standard, introducing new topics, adjusting some and removing others. These changes requires the educational landscape to be adjusted: teachers' knowledge needs to be adjusted as well any support resources such as textbooks, software, or even the mere writing tools. A culture around the original ideas needs to be built.

We thus ask ourselves the following research question: What are the paths between the emergence of a scientific idea to its applications in the classrooms? We hope this example will be useful to others as they change, or respond to changes in, curricula. Our interest lies in the mathematical knowledge with a special interest on the evolution of the mathematical notations, this set of customs that forms the carrier of the formal knowledge of mathematics. This question is important when planning learning resources such as intelligent tutoring systems, quiz systems, or simple works made of multiple web-pages. Only with this in mind is it thinkable to receive the mandate to create teaching resources for a subject where little teaching experience is available. Such an issue arose in England in 2014, when teaching computing, including programming, suddenly became compulsory for ages 5–14, and teachers were confronted with mandates from management such as "you teach Powerpoint, now teach Python".

Many domains of mathematics have seen an introduction within the last decades, most promoted by the steadily growing need of the technology industries. As simple examples, one can mention the general domain of analysis, with the focus to calculus now broadly taught across many mathematics secondary schools (but only introduced at Harvard in 1738 [15]), and still controversial in USA in the 20^{th} century [26]). One can also mention probability and statistics whose teaching in schools was patchy in the U.K. at least in much of the 20th century [5]; in France [24] and many other countries this domain was only introduced at the turn of the millennium. Both these broad fields introduced new teaching challenges. Both contain a large body of mathematical literature, are connected to many other topics of the mathematics, and can be supported by pedagogical knowledge. Their development uses a language of notation which subtly diverges at least along languages. Instead of exploring these large domains, we prefer to restrict to a theory which is smaller and probably has a more homogeneous representation: set theory. This also has a distinctive notation (unlike, say, Data Science), which makes tracking its evolution easier.

1.1 Related work

While several research efforts have been made into the history of the curriculum standards (e.g. [6] or [10]) we know of few works that have employed the history of notation to relate the history; most notably the work of studying the evolution of symbols on mathematical concepts [18]. We also observe that many mathematics learning tools are rooted in particular historical, pedagogical, and national contexts and propose to use the history to take a birds' eye perspective. Curriculum standards debates have been often bound to various political and influence attempts as demonstrated in [30]; our case study looks at the content and knowledge in a much more factual fashion.

1.2 Paper's organization

Our paper is a case study on the introduction of set theory. It first depicts a few essential qualities of a mathematics concept so as to make it into a teachable concept. Based on this characterizations, which show the cultural roots of a concept, a precise historical walk is depicted for some of the elementary concepts of set theory. The paper then envisions alternative ways that the concepts could have grown with and the probable implications for current notations in current communication systems.

1.3 Decomposing a Mathematics Concept

The introduction of a mathematical concept among the learning objectives can have multiple facets which we try to enumerate here: We describe essential elements of a mathematical concept. A mathematical concept is, here, any formally defined idea that allows mathematical operations and has mathematical properties, for example the concept of matrix (2-dimensional arrays of numbers), of limit of a real function (an operation on real functions to evaluate approaches of its values), of set defined by a condition (a notation to describe formally a set by expressing a condition on its members), or of maximum likelihood estimate (a technique to estimate the parameter of a supposed probability distribution).

Typical facets of a mathematical concept include the following:

- Name: How is it named? how does it decline itself in the various grammatical forms?
- Definition: How is it defined?
- Properties: What are its properties? (proven or suspected)
- Connections: How does it connect to others?
- Notations: How is it graphically represented as notation?
- Properties and Connections Notations: How are the properties or connections represented? (how can you compute with it?)
- Applications: What can you apply or deduce from it, or with its help?

These facets are important for the sole objective of understanding the concept. An example description of a mathematical concept could be the Wikipedia page `https://en.wikipedia.org/wiki/Complement_(set_theory)` or the classical book on set theory [13, p. 17] defining the complement of a set.

While one may expect the mere definition to be already the essence of the concept, these facts already show how much connected to other mathematical concepts a mathematical idea is: The concepts, be them presented in original article forms or as an introductory summary, all live within the culture of mathematicians of their times which encompass language (names, declinations), knowledge (definitions, properties...), arts (in particular, graphical representations), customs (ways of saying and of noting), capabilities (acceptable and not acceptable operations), and habits (classical resolution steps, ...).

When they are explained or presented, they do so in a way that is connected to a network of concepts; these can be represented by references to other scientific works but also other more digestible references. It is not rare that a concept explanation becomes refined and reformulated. Textbooks and articles that describe the concept without the claim for originality can offer refinements and iterations, distillation. This helps better connections, easier proofs, better operations through notations. This maturity process can be well perceived for the arithmetic operations when reading the book [18] which explains, in particular, the impact of notations to empower the conceptual thinking.

1.4 Teaching a Concept

As every piece of knowledge, a mathematical concept can become the subject of teaching: By a teacher on a blackboard, within a digital communication channel or by fully automated tutorial software: All these learning methods employ similar facets of the mathematical knowledge concept.

Before a concept can be taught, an even longer maturation process happens. The first step is in recognizing its importance for currently relevant societal goals. As of this writing, this happens strongly with the many linear algebra and analysis processes behind machine learning: The need for mastering the mathematics behind machine learning algorithms and possibly devising new applications for many trained engineers is currently strong (a simple example could be seen in [17]).

Earlier needs can be interpreted such as the need for modular arithmetics around multiple computer science courses to support encryption or the need to calculate more precisely that stimulated Charles Babbage and Ada Lovelace to create the Analytical Machine.

The second interconnection step relies on the pedagogical preparedness of the concept: Depending on the amount of evolution the concept has gone through, it can take many decades before it becomes applicable in teaching situations. Teaching a concept involves that the teacher or software is able to represent and operate on the concept but can also react to many correct or erroneous representations or operations done with it. This requires the software or teacher to connect far broader realities than that done in the applications or properties proofs of the concept. The example situation of Sean's even numbers' definition in [2] and its articulation of the *mathematical horizon* show well how much broader the scope of teaching is than that of proving or applying is.

This maturity level is anchored in the same cultures but is enriched by the more education oriented cultures: The fact that the concept is manipulated in classrooms means that many alternative ways to think of it, to express it, to manipulate it, and to operate with it will be thought of. School teaching as well as educational software are two representations of the theory which require more completeness than just explaining the concepts in an article or book: If considering the culture around a concept as the set of ways to express and manipulate a concept, including how to write it, how to read it, how to input it, how to operate on it, then both educational software and teaching practice connect to a much broader culture than the mere expert explanation or even simple textbooks.

The risk of minimizing this extra culture and to consider that teaching only requires the application of the core knowledge is non-negligible as the introduction of new topics posits that the teachers master the topic. An example situation has been told us as a story of parentship when a 10 years old child in Ontario (Canada) was required to count the probabilities of the sum two regular cubic dice: The teacher, then, suggested to use a table of all two dice values but counted the double matches as occurring in one sense and the other (so, to get the sum of 4, one would use the pairs $(1,3),(2,2),(2,2),(3,1)$. Interrogated by the parent, the teacher's answer was: We let the vote decide if we should count double matches twice (this vote was done in the classroom and when learning the concept at the teacher's university) [22]. This story shows well how much challenged a teacher can be and how much non-mathematical the knowledge can be developed: Beyond the knowledge impact, the essentially non-proof-based approach used by the teacher may lead to ill-founded mathematical assertions being used in the children's future.

2 History of a concept introduction: the set notations

Set theory, and the notation that goes with it, are engrained in (university-level) mathematics today. But it was not always so.

2.1 The theory of set theory

> Set theory, as a separate mathematical discipline, begins in the work of Georg Cantor. One might say that set theory was born in late 1873, when he made the amazing discovery that the linear continuum, that is, the real line, is not countable. [1]

Hence the subject is essentially 150 years old. But it did not spring fully-formed from Cantor's head: We still had Russell's paradox [23] and others to deal with. The first abstract formalism of this set theory (though we should not ignore the works of Boole [3] and their graphical development in the form of Venn diagrams [28]) appeared in 1908 [31], but Zermelo–Fraenkel Set Theory as we know it is barely 100 years old [11].

The development of the notation did not wait for the abstract formalisms to be developed. The notations $\cap$ and $\cup$ were in use in 1888[3], and [21, p. V] introduced a set membership symbol, using a regular ϵ rather than today's $\in$.[4] An example of early notation is in figure 1.

quae eodem tempore sunt a **et** b; $a \cup b$ **est classis individuis constituta qui sunt** a **vel** b.

47. $a, b \,\epsilon\, \mathrm{K} \,.\, \supset \therefore x \,\epsilon\,.\, ab :=: x \,\epsilon\, a \,.\, x \,\epsilon\, b.$

48. $a, b \,\epsilon\, \mathrm{K} \,.\, \supset \therefore x \,\epsilon\,.\, a \cup b :=: x \,\epsilon\, a \,.\, \cup \,.\, x \,\epsilon\, b.$

Fig. 1. An early presentation of the concept of inclusion of set theory in [21, p. XI].

However, these notations were essentially confined to logic and set theory, and did not greatly impact "mainstream" mathematics. For example, the second author's father, who went to university in the 1920s, did not naturally use the

[3] [20, §2, pp. 1–2]. According to [4, Vol. 2, p. 298] this was the first definition.

[4] `https://mathshistory.st-andrews.ac.uk/Miller/mathsym/set/` is slightly surprised at this, and the fact that Peano's existential quantifier was a rotated (the original says "backward", but this is impossible with 19th-century printing technology) E *with* serifs. But all Peano's "new" symbols were existing characters being used in unusual ways, rather than asking the printer to make new symbols. For example [21, p. XI] uses a rotated C for "subset".

notation. It was probably the Bourbaki texts that popularised these notations in mathematics outside set theory/logic. Apparently these texts also introduced the $\notin$ symbol. From this point, we can say that set theory notation became the notation of mathematics.

2.2 School-level set theory

In the 1960s, under labels such as "New Mathematics", the notation (at least $\cup$, $\cap$ and $\in$) and the elementary use of set theory was introduced into experimental school curricula, at least in U.S.A. and U.K. Gradually, more curricula would adopt these, but often as options. This would make life difficult for university teachers: the second author often encountered classes where roughly 2/3 had encountered these symbols, but 1/3 had not. It wasn't until 2017 that their teaching was required[5] (for those 16–18 year olds studying mathematics) in England [9, OT 1.3]. The precise set of symbols required was $\cup, \cap, \in, \notin, \subset, \subseteq$, $\{\ldots\}$ and $\emptyset$.

In French speaking Europe, and a few other countries, "Modern Mathematics" (mathématiques modernes) were introduced in the 1960s too. It had as most important premise to make set theory the basis of most other mathematical concepts starting as early as the age of 7. But the introduction was so radical that it was criticized for the social inequalities it created and demised less than 20 years later [6,10].

It is worth noting that, just as Venn diagrams preceded $\cup$ etc. chronologically, they do pedagogically: the same reforms to mathematics teaching in England that introduced the symbols for 16–18 year olds made Venn diagrams part of the curriculum for 11–14 year olds [8, KS3], Venn diagrams and mappings with them were made at around the age of 11 in modern mathematics.

We can see a similar development in the most widely spread softwares used for calculations such as computer algebra systems: those of the 1960s and 1970s (CAMAL, Macsyma, Reduce etc.) do not have "set" as a built-in concept, whereas those of the 1980s (Maple [12, pp. 5-6], Mathematica) do. Reduce had set functionality added in the the early 2000s. Computer algebra systems are the standard tool used to perform mathematically correct elaborate calculations in integrated mathematics teaching systems that aim at least at the secondary school level at least for their ability to perform symbolic evaluations of the students' answers.

2.3 Possible Causes

There seems, unfortunately, to be little discussion as to why set theory has been kept away from school that long. We can conjecture that the difficult paradoxes

[5] The second author was a member of the committee that made this decision (formally, a recommendation to Government), and its principal proponent. It succeeded as part of a more general move to reduce the number of options and increase the compulsory core.

that a teacher can fall into are an undesired trap. For example, Russell's paradox, which illustrates classes that should not be sets, requires, to be discussed, a very clear set of axioms.

We can also conjecture that set-theory was not prepared sufficiently in its graphical expressions with reference works such as [13] having ignored Venn diagrams whereas this notation is the essential representation deemed useful in school books. This gap between theory and teaching is very visible in figure 2.

words, g is a function from Y onto the set X/R of all equivalence classes of R. The function g has the following special property: if u and v are distinct elements of Y, then $g(u)$ and $g(v)$ are distinct elements of X/R. A function that always maps distinct elements onto distinct elements is called *one-to-one* (usually a *one-to-one correspondence*). Among the examples above the inclusion maps are one-to-one, but, except in some trivial special cases, the projections are not. (Exercise: what special cases?)

To introduce the next aspect of the elementary theory of functions we must digress for a moment and anticipate a tiny fragment of our ultimate

Fig. 2. Two very different representations of the concept of one-to-one mappings: on the cover of the exercise leaflet of [7] (for pupils of about 8 y old) and among other concepts in the classical set-theory book [13, p. 32] (for scientists).

3 Alternative Ways Set Theory Could Have Been Represented

From the long history of the concepts of sets, more than 150 years, one can see that many ideas could have emerged being different. We propose ideas that indicate possibly different evolutions.

In terms of notations, alternative symbols could have been used: A more Venn-diagram-like notation could have been used as in the following equation. It is likely that the typesetting difficulty prevented this:

$$\boxed{4n|n \in \mathbb{N}} \cap \boxed{3n|n \in \mathbb{N}} = \boxed{12n|n \in \mathbb{N}}$$

Taken even further, the intersection symbol could have been presented by intersecting the rounded boxes as in the following:

$$\boxed{4n|n \in \mathbb{N}}\boxed{3n|n \in \mathbb{N}} = \boxed{12n|n \in \mathbb{N}}$$

And a similar graphical construction could be invented for the union operation. Such an alternative notation can be fully formalized. Learning software makers or animation makers could, nowadays, with such a notation, offer a much smoother transition between the formal statements such as above and the Venn diagram representations that have been widely used at schools thus becoming a much more easily rememberable concept.

Many alternative ways might be more compatible with other approaches. For example the first set above can be written as in $\{4n|n \in \mathbb{N}\}$, $\{k|\exists n \text{ with } k = 4n\}$ or simply using ellipses $\{0,\ 4,\ 8,\ 12,\ 16, \ldots\}$. Each of these representations is in use but provides different ambiguities or abilities to solve. As can be seen in this StackExchange question [16], the desire for richer learning materials in set theory still remains and few stabilized educational studies have been realized as shown in [19].

Explanations of concepts such as the infinite sets as in the video [29] also provide example of expressions using a much broader vocabulary that expresses sets and their operations with graphical animation methods. The popularity of the 3-blue-1-brown video series [25] shows that animations to explain mathematics can impact the understanding of many. This new set of expression means impact the mathematical cultures deeply: new writing methods need to be created, new ways of consumption are expected, new notations are likely to appear. Learning software projects are all subject to this evolution and may be carriers of new such expressions faster than school books have developed. This implies that it may become the role of learning software vendors to innovate in the introduction of notations.

4 Potential Learning Processes for Future Concepts

New mathematics ideas flourish, with new approaches to manipulate the mathematical concepts. The community of mathematicians is active in creating new concepts or rewriting existing ones and it will grow. In this paper, we have provided the example of a long-winded set of mathematical concept which went from mere ideas into a practice that will be soon teachable by tutoring systems. We have sketched evolution of the cultures around the concept so as to demonstrate how much transformation and additions a mere scientific idea can need to become teachable. It is likely that such processes appear further.

Among the current evolutions of mathematical knowledge, algorithms are having greater roles; this includes several proofs but also includes algorithms for mundane tasks such solving equations or calculating optimal values and let computers do a part of that algorithm with trust. We contend that this, again, will need different forms of expressions so that proofs can be exchanged and learning resources can be built.

Two trends are especially affected by the evolution of technology: digitization and internationalization:

While some mathematicians still consider they should only want to express mathematics with a pen-like device. Digitally-expressed mathematics is there for learners and for practicing mathematicians. It brings other affordances (e.g. distance learning) and an amount of simpler and more expressive ways of explanation. As an illustration, a typology of *mathematics augmentations* has been started so as to stimulate the field [14]. It is clear that animations will help connect operations with geometric perception and that more such techniques will fuel the knowledge.

On the other side, the need to support *wandering readers* will grow: far from the sustained participation to a learning channel that pupils have in regular classes, we contend that users will evolve with even more cross-navigation actions: they will navigate to particular places in websites or open books at a definite page following an index. This brings a bigger visibility to the need for referencing topics and for an international exposure. Similar to some established multilingual efforts such as [27], culturally correct translations will be expected and counted upon, including taking in account all notational conventions. While, as could be seen with set-theory for which the emergence we reported has used at least four languages (German, Latin, Italian and English), the language is not so important for a scientific elaboration, the language is, on the contrary, an important cognitive factor for wandering readers who expect as much as possible a closeness to their own (notational, language or conceptual) cultures.

These two trends may bring considerable strengths in bringing mathematicians' concepts, that are so much needed for the development of societies, faster into learning resources. We contend, further, that they may help bringing more subject specific exchanges and web a more dense mathematical horizon knowledge.

References

1. Bagaria, J.: Stanford encyclopedia of philosophy: Set theory (2023), available from `https://plato.stanford.edu/entries/set-theory/`.
2. Ball-Loewenberg, D., Bass, H.: With an eye on the mathematical horizon: Knowing mathematics for teaching to learners' mathematical futures. In: Paper prepared based on keynote address at the 43rd Jahrestagung für Didaktik der Mathematik (March 2009)
3. Boole, G.: An investigation of the Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities. Macmillan (1854)
4. Cajori, F.: A History of Mathematical Notations. 2 vols. Open Court (1928), browsable from `http://www.archive.org/stream/historyofmathema031756mbp`
5. Conway, F.: Statistics in Schools. Journal of the Royal Statistical Society. Series A (General) **149**, 60–64 (1986)
6. De Bock, D.: Modern Mathematics: An International Movement Diversely Shaped in National Contexts, pp. 1–9. Springer International Publishing, Cham (2023). `https://doi.org/10.1007/978-3-031-11166-2_1`, `https://doi.org/10.1007/978-3-031-11166-2_1`

7. Denise, H., Denise, J., Polle, R.: Mathématique CE2. 3 cahiers. Delagrave (1972), cover picture from https://rs-genea.com/omeka/items/show/2590 (checked 2024-06-15).
8. Department for Education: National curriculum in England: mathematics programmes of study. https://www.gov.uk/government/publications/national-curriculum-in-england-mathematics-programmes-of-study (2013)
9. Department for Education: Mathematics AS and A level content April 2016 (Content for mathematics AS and A level for teaching from 2017). https://assets.publishing.service.gov.uk/government/uploads/system/uploads/\protect\penalty\z@{}attachment_data/file/516949/GCE_AS_and_A_level_subject_content_for\protect\penalty\z@_mathematics_with_appendices.pdf (2016)
10. Dessauw, G.: Mathématiques: 3 les mathématiques modernes, available in the collection "Le temps des instituteurs" from http://www.le-temps-des-instituteurs.fr/ens-maths-modernes.html (checked 2024-06-15).
11. Fraenkel, A.: Zu den Grundlagen der Cantor-Zermeloschen Mengenlehre. Mathematische Annalen **86**, 230–237 (1922)
12. Geddes, K., Gonnet, G.: MAPLE User's Manual. Tech. Rep. CS-81-25 University of Waterloo, MapleSoft Inc. (1981)
13. Halmos, P.: Naive Set Theory. D. Van Nostrand Company (1960)
14. Head, A., Xie, A., Hearst, M.A.: Math augmentation: How authors enhance the readability of formulas using novel visual design practices. In: Barbosa, S.D.J., Lampe, C., Appert, C., Shamma, D.A., Drucker, S.M., Williamson, J.R., Yatani, K. (eds.) CHI '22: CHI Conference on Human Factors in Computing Systems, New Orleans, LA, USA, 29 April 2022 - 5 May 2022. pp. 491:1–491:18. ACM (2022). https://doi.org/10.1145/3491102.3501932, https://doi.org/10.1145/3491102.3501932
15. Holton, G.: History of the department (2014), available from https://www.physics.harvard.edu/about/history
16. Liu, A.: Fun set theory for kids (2020), available from https://matheducators.stackexchange.com/questions/18267/fun-set-theory-for-kids .
17. Manning, C.: Stanford cs224n: Nlp with deep learning || winter 2019 || lecture 2 – word vectors and word senses (2019), available from https://www.youtube.com/watch?v=kEMJRjEdNzM, checked 2024-05-23
18. Mazur, J.: Enlightening Symbols: A Short History of Mathematical Notation and Its Hidden Powers. Princeton University Press (2014)
19. Narli, S., Baser, N.: Cantorian set theory and teaching prospective teachers. International Journal of Environmental and Science Education **3**(2), 99–107 (2008)
20. Peano, G.: Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann: preceduto dalla operazioni della logica deduttiva. Fratelli Bocca (1888), available from https://matematicaitaliana.sns.it/opere/138/.
21. Peano, G.: Arithmetices principia: Nova methodo exposita. Fratres Bocca (1889), available from https://archive.org/details/arithmeticespri00peangoog/page/n12/mode/1up .
22. Pinsonnault, M.: Private email: compétence en nouveau domaine, message with ID CDC6AF88-3BAD-45A3-BDA2-D56654796A0A
23. Russell, B.: Letter to Frege (1902). In: van Heijenoort, J. (ed.) From Frege to Goedel : a Source Book in Mathematical Logic, 1879-1931, pp. 124–125. Harvard University Press (1967)
24. Régnier, J.C.: Enseignement et apprentissage de la statistique: Entre un art pédagogique et une didactique scientifique. Statistiques et Enseignement **3**(1) (2012)

25. Sanderson, G.: About 3blue1brown (2024), available from https://www.3blue1brown.com/about (checked 2024-05-24).
26. Swenson, J.: Calculus in the High School. Junior-Senior High School Clearing House **5**, 347–349 (1931)
27. Upsing, Britta; Rittberger, M.: The translator's perspective on translation quality control processes for international large-scale assessment studies. Translation & Interpreting **10**(2), 55–72 (2018)
28. Venn, J.: On the diagrammatic and mechanical representation of propositions and reasonings. Philosophical Magazine Series 5 **9**, 1–18 (1880)
29. van Waerebeke, D.: Voyage au pays des maths: Sur la route de l'infini, published by ARTE, available from https://www.arte.tv/en/videos/097454-005-A/the-land-of-mathematics/.
30. Wright, P.: The math wars: Tensions in the development of school mathematics. For the learning of mathematics **32**(2), 7–13 (2012), available from https://www.jstor.org/stable/23391957.
31. Zermelo, E.: Untersuchungen über die Grundlagen der Mengenlehre, I. Mathematische Annalen **65**, 261–281 (1908)