On the Specification of the Relevant Mathematical Notations

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Abstract. Mathematical notations are a form of graphical language used to express statements. The language is thought of as formally well defined and is made of a broad spectrum of symbols. Because of its formality, it is common to consider that mathematical notations (or the mathematical knowledge) are universal and can be universally understood. This consideration is not correct. We describe examples of notations which fail to be universal and propose explanations for these failures. Thus, the following question is natural: How or who should specify the relevant mathematical notations when planning a content or software project? We propose approaches to answer this.
The breadth of the spectrum of mathematical symbols, and the dominating handwritten practice, brings another important aspect to this question: Every mathematician considers himself or herself empowered to create new notations. We explain why this freedom of invention is important and sketch implications for the design of educational tools.

Keywords: Notations· Mathematics· Language· Cultures · Invention

1 Motivation

Mathematics is a branch of science that makes it possible to operate deductions, which include calculations, so as to conclude statements from other statements. Based on a very small set of axioms and the strict application of logics, mathematics has become the central tool to model reality in most applied sciences. Applied sciences are becoming broader every single day and are touching every day lives more and more. As a result the need for every human to master mathematics grows.

Mastering mathematics means to have learned it. In turn, learning mathematics involves not only mastering the knowledge but being able to perform new deductions and apply them. The deduction steps are, nowadays, performed mostly using the symbolic language of mathematical notations which is learned together with the concepts.

Mathematical deductions or models are exchanged via communicative acts: lectures, papers, emails, blogs, social networks... The persons participating to the communicative act all should understand the mathematical notation used if they want to fully understand what is being told. Thus it is natural to study the set of mathematical notations within the environment of a communicative act.

Learning systems are examples of such environments: through a user-interface, the learning system uses mathematical notations and requests them to be manipulated or input by the learner; thus mathematical notations are a central ingredient of these learning systems.

When one considers the very classical environment of a mathematics classroom, the flow of information is mostly well defined: Teachers are responsible for the content being viewed, presented or written, for the use of notations, for the concepts being learned and he or she knows that it is doable to do that in a way that the students will learn.

The more scattered learning situations met in the research activities or in advanced mathematics learning, for exemple when reading the documentations of statistical software, are examples of environments which must establish a set of notations and possibly need to explain it. This is probably the reason why ISO-8000 [1] has emerged as a description of “the norm” of notations. However, the content of such a standard is not how mathematics is being learned: many persons who visit such environments will likely have to adapt the notations they have learned to the notations being used; this may be cognitively demanding and thus may slow down the reading.

Because learning software needs to teach mathematics to students “where they are”, we ask the natural question: How to decide and specify the set of notations for mathematics learning software?

2 Definitions

We describe as mathematical notations the set of graphical constructs used to describe concepts of the mathematical science using other means than words. In general notation is written in horizontal lines (from left to right or right to left) with the addition of vertical layout constructs. Sometimes complex graphical systems are built with mathematical notations. In many cases, mathematical notation is expressed together with sentences and replaces, grammatically, a word or a clause. This makes it possible to read them aloud.

Mathematical notations are made of symbols, which represent a construct of a well-known semantic. For example an operation (e.g. the and operation of two predicates as in $S \wedge T$), a digit (e.g. the number 7), a decoration (e.g. the bar on top of a statistical sample denoting its average value as in $\overline{M}$). Each of these notations are examples of ambiguous notations: They only have a meaning if the context is clear (e.g. $\wedge$ could also mean an alternate product of differential forms, the number 7 does not indicate if it is a natural number of a number modulo 5, $\overline{M}$ could also be the complement of the set M).

Each symbol comes with a set of rules to compose it in a meaningful way for example the intersection can be taken between two sets; but intersecting functions or numbers makes little sense to most persons.

The idea of mathematical notation does not define how they are understood. It is the role of an explanation of the notation to do this and to explain the rules of composition. These rules are often coupled with further reasoning that mirror the properties of the concept in visual effects observed on the notations.

It is common practice to represent the structure of mathematical notation as a tree, where one would see operators applied at the root of a branch and their operands as their children. This representation helps approach a machine based semantic of the mathematical operation as is done in MathML-content [9, chap. 4] or [7]. This bridge to the semantic representation of formulæ is an aspect that can empower learning, computing, and communication systems and support the generation of multiple notations from the same tree. We shall only speak little about this representation preferring to focus on the notations aspect.

3 Rationales behind Mathematical Notations

For a person speaking and reading, at least with a Western language, mathematical notation is an intuitive way to represent mathematical concepts. Compare the following expression in words "the derivative of the product of two functions is the sum of the product of the first function, derived, times the second, and the product of the first function times the second, derived." with "$(f \cdot g)' = f' \cdot g + f \cdot g'$". It is rather natural to use the second expression and, at most, hear the first. Two other cases of strongly different expressions are in figure 1.

Thus, mathematical notations has a compression and memory effect. However, this memory effect is only working if the rules that one knows are satisfied. Indeed, an experiment [16] where participants were requested to remember parts of the formulæ showed a clearly stronger ability to remember the formulæ that were well formed compared to those that did not match the well known structure.

Mathematical notations is explained in [20] to have been fuelling the science and understanding of further mathematics: Among the most important notations of human history is the ability to operate and think with numbers. This ability is largely fed by the notations of numbers: It is the central carrier to perform operations such as long additions, subtractions, multiplications or divisions. The evolution has, however, led to diverging notations for each culture.

The decimal representation of numbers is one of the simplest diverging notations: 1.732 can be understood as an integer between one and two thousands (e.g. when articulating a number of Euros in French) or an approximation of the

Boyle's Law: The absolute pressure exerted by a given mass of an ideal gas is inversely proportional to the volume it occupies if the temperature and amount of gas remain unchanged within a closed system.

$P \propto \frac{1}{V}$

Pythagoras Theorem: the area of the square whose side is the hypotenuse is equal to the sum of the areas of the squares on the other two sides

$a^2 + b^2 = c^2$

Fig. 1. Two statements expressed in words or in symbols

square root of three (e.g. in the English language). While this difference seems easy to compensate by providing external context, encountering such characters somewhere random on the web may prevent an immediate understanding as extra information is needed.

The expression of mathematical notations has been chosen for its diagrammatic expressivity. As an example, the horizontal alignment, one after the other, of the terms of an operation allows to represent the application of the associativity property (as in $f \circ (g \circ h) = (f \circ g) \circ h = f \circ g \circ h$). Similarly, when on both sides of an equal sign, additions can be visually transported by "taking them on the other side" (changing their sign). This kind of operation is, likely, reminiscent of the geometric addition of lengths when connecting several segments behind each other. Characteristics of the operation such as the ability to move terms (move them around, to the other side of the equality...) are learned when the operations and their notations are learned. Many of the properties can be expressed in a way that corresponds to a move that is recognizable.

We call "haptics of notations", the connection between the graphic configuration of notations and the moves that are possible. They are an integral part of the understanding of the concepts and of the notations. As far as the author can tell, the movements of the mathematical notations are mostly learned simultaneously as the concepts themselves. An easily remembered example is the distributivity property in figure 2

The effects of allowing manipulations to support learning of mathematics are numerous. The subject of "manipulatives" is a rich object of software and didactical developments. It goes as far as "embodied cognition" described, for example, in this paper [24]; some of the effects include a better memorization.

Manipulations allowed by mathematical notations are helping mathematical habits of mind which are described as differentiating the experienced mathematicians from the less experienced ones [6].

To conclude, we cite the motivating introduction to algebra presented in [13], which shows well the intimate relationship between manipulations and understanding: "*Performing calculations with them and using the results to make predictions requires an understanding of relationships among numbers. In this chapter, we will review sets of numbers and properties of operations used to manipu-*

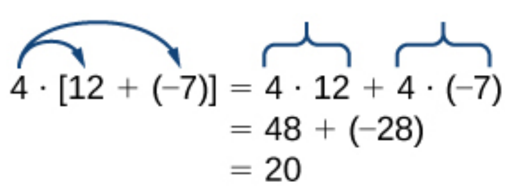


Fig. 2. A *haptic* representation of the distributivity property of the multiplication on the addition from [13, Chap 1.1].

late numbers. This understanding will serve as prerequisite knowledge throughout our study of algebra and trigonometry."

4 Where do Notations Live

Having described the rich aspects of knowledge that are connected to mathematical notations, we now try to describe the learning processes of mathematics where notations are learned typically along the lives.

First concepts of mathematics appear with fingers and rythms and in geometry from the age of 2. The surroundings of the children are essential in their structuring of this understanding process.

Mathematical notations start to be learned with numbers. First till 10 or 20 (first years primary), then bigger. The first differences accross the cultures start to appear already here: E.g. in Germany, the first two years only show the children numbers till 20 whereas in France, the numbers till 100 appear almost in the first year. Written operations start from 9 or so, unknown and polynomials about at the age of 14 and proofs start to be explicit at the age of 15. But divergences have already started then.

A very visible variation between cultures appear in the 17 different ways of performing the long division as reported in MathML-long-division [9, Section 3.6.8.3].

Multiple other smaller differences occur, in particular those fueled by the language changes. For example, the greater common divisor is written using the first letters of the words in each language (ggT in German, gcd in English[1]) or even some other influence (in French, it can be used as pgcd or pgdc depending on the modernity claims of the authors).

Mathematical practice is, itself, influenced by the cultures: When in France a proof can never be summarised in a picture (and is, instead, based on a formal argumentation chain expected in words or symbols); in Germany a proof is often best explained using a picture [17]. Mathematics is tightly bound to the possibilities of languages as explained in this citation: "we bring mathematics into existence by talking about it, and the way we talk about it changes the questions we can ask" ([5] cited in [12] which goes deeper in the affordance of understanding mathematical concepts based on grammatical differences).

Moreover, it should be noted that mathematical understanding and mathematical notations can be influenced by the semantic fields induced by the surrounding languages. For example the concept of "Takt" in German means the physical notion of frequency but it also means the unit of measure of music notations, it may also mean the empathic behaviour of a person. No word exists with these three meanings in French or English (for example) thus the memory of mathematical concept may be connected to different memories depending on the language of the learning.

[1] See `http://notations.hoplahup.net/Census/CD-arith1/gcd` for more notations of the greatest common divisor. This website contains multiple other notation divergences examples.

The impact of cultures on mathematics learning and operating has been so deep that an entire research movement has emerged called *ethnomathematics* which aims at reinventing mathematics having acknowledged the very deep role of cultures, both methodologically and conceptually. An example of reflections is presented in [3].

Based on these early divergences, mathematical notations becomes scattered to a point that ambiguity can, sometimes, only be solved by long investigations. Ambiguities such as the following exist:

- The tradition of using red color to denote negative numbers and green for positive ones (in Western cultures) is inversed in Japan.[2] A misinterpretation may be the basis of important business decisions.
- Ambiguities include the use of the vertical bar to mean absolute value of a real number, the determinant of a matrix, the cardinality of a set, the such-that construct of sets, or the conditional probability. They imply, for example, that systems that read aloud may encode heuristics such as: $|M|$ is a determinant but $|a|$ is an absolute value (depending on the case of the argument).
- The meaning of $(-100, 100)$ in a course of differential calculus in English for a reader that has not read the course start to end will not be clear: It can be interpreted as the coordinates of a point ($x = -100$, $y = 100$) or an open-interval of the numbers between -100 and 100 not included.

This section has shown us that aspects of the environment around mathematical notations in a communication act allow a person to "enact" many possible semantic behind mathematical notations when it is not provided with a complete context.

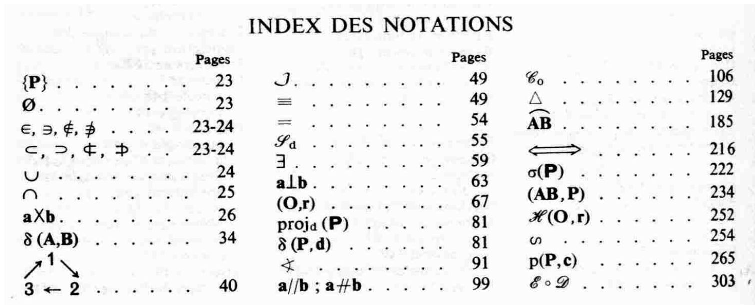
As a way to make sure that the possibly new mathematical notations introduced in a book are clear, tt has been common for mathematical texts to include a list of notations at the end of their works. Such a table of notations is presented in figure 3.

5 Inventions of Notations

In his activities with various authors of mathematical texts, the author constantly met the freedom of mathematicians that inventing notations is a "fundamental right": It should be normal for any creator of mathematical documents to be able to explain enough so that readers can be endowed with new notations.

While it seems presumptuous to express this as a fundamental right, it should be noted that history has repeatedly seen the invention of new notations: The detailed explanations of the symbols' evolutions of [20] demonstrate this well as the invention of symbols has made it possible to understand new concepts such as arithmetic operations or the mere number 0. Another evidence stems from

[2] This can be seen in the first web pages of the Tokyo and New York stock-exchanges: `https://www.jpx.co.jp/english/` and `https://www.nyse.com/trading-data` .



INDEX DES NOTATIONS

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$\mathbf{a} \times \mathbf{b}$	26	$(\mathbf{O}, \mathbf{r})$	67	$\mathscr{H}(\mathbf{O}, \mathbf{r})$	252
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		$\measuredangle$	91	$\mathscr{E} \circ \mathscr{D}$	303
		$\mathbf{a} // \mathbf{b}$; $\mathbf{a} \# \mathbf{b}$	99		

Fig. 3. The list of notations at the end of a book on geometry in French [11].

the notorious reference of F. Cajori [8], recognized as the most complete history of mathematical notations: It contains words such *invent* at least once every 5 pages.

As is the case with just any concept, its relevance only occurs when it is adopted by others and becomes a reproduced *habit of mind* [6]. This can explain why the invention of notation is rarely celebrated as an important action while the concept is more often attributed. Some notations carry their inventor's name (e.g. the derivative notation of Leibniz or Newton, the δ of Kronecker or even the Einstein convention).

The need to invent mathematical notations may be doubted of. After all, the body of knowledge of mathematics is rich and mature. However the will to understand even more mathematics is steadily growing. Thus [15] presents in its motivation the ever-growing need for more understanding of data science. In this field elementary ambiguities as simple as f_X remain: When just found in a web-page describing physical movements, f_X is likely to mean the derivative of the function f in the direction of the vector field X; if the page describes statistics, it is likely to mean the probability density function of the random variable X. What happens if found in a page that describes the statistics of the movement of particles in the air? It is not clear yet. The personal experience of the author in teaching probability and statistics has shown that the notations of these two neighbouring fields, which also use linear algebra, geometry, and analysis, would be much less confused if it were reinvented.

6 Implications for the construction of learning softwares

Having shown a glimpse of the breadth of the diversity of mathematical notations, we turn to the need of software developers to be able to represent the mathematical notations to express mathematical knowledge in a way that allows operationalisation, e.g. when planning a content project or a learning software.

For example, we expect that makers of computing systems to organize how they define mathematical notations in such a way that both the notations displayed in results and the notation used for input look similar, e.g. that the look of buttons is the same as in the rendered formula.

- A first approach is to hard-code notations: Examples of one-size-fits-all applicants are numerous. Pocket calculators have neglected the difference of decimal point, accepting that the period "." is universal, and mostly do so for trigonometric functions. Similarly, the formula input with a pen in MicroSoft Windows introduced with Windows 7 is an example of a tool that has been designed with the western culture in mind and, according to his author [22], would require a complete retraining if it wished to be adapted to another culture. We claim that the cognitive costs of translating the mathematical notations to that of a different culture (mostly, the Anglo-Saxon culture) is similar to the cost of entering the formulæ in some programming language, thus it is natural to expect a learning system to adapt to its learners' cultures [21].
- Addressing few languages may be a small start: Wolfram|Alpha[3] has been introduced in 2009 with a very broad set of classical mathematical tasks; contrary to the claim of its creator in 2012 [25] to go international soon, it only has introduced other languages very slowly (Japanese 2018, Spanish 2022, see [26,4]): Both these pages show that new languages are the result of long adaptation processes and go further than mathematical notations.
- More advanced solutions exist offering multiple languages: Systems of translations between formulæ encoded in an abstract format and their notation-language (e.g. MathML-presentation or LaTeX) can be used by mathematical software systems. They allow the abstract format (e.g. OpenMath which can be both calculated upon and rendered). Ideally they also allow them to be input. To perform these translations, a complex set of rules can to be maintained; examples include [19,18,2], it is the set of notation-rules each of which associates a pattern in the OpenMath tree with a corresponding rendering in MathML-presentation. These sets of rules may also include the user-interfaces buttons of an input; the preferability of some symbolic constructs compared to others (e.g. to facilitate the recognition or to include the button in a more readily accessible place than the other). This approach is, to our knowledge, the only possibility to encode notations abstractly and completely.

How many sets of notations should be maintained? One first approach to decide on the amount of notations, is to create a set for each language: This set must cover all operators and may be very resembling to others but must be in a position to take in account the adaptivity for the numbers' format (a normal expectation of users) as well as classical symbols.

Learning software makers may take the variation further to ensure that the cognitive load of users remain devoted to their learning. Each of the symbols used should be screened for their relevance for the target audience. This is probably

[3] Wolfram—Alpha is a recognized search engined for mathematical questions. See `https://wolframalpha.com` .

best done by content authors with teaching experience in the field. For example, a math teacher would be able to identify, depending on the age, if the symbol for division of two numbers should be as a fraction (as in $\frac{24}{3}$), using colon (as in $24 : 3$) or using the $\div$ sign (as in $24 \div 3$) (the age of the target user should be the decision basis) or if the root of -1 should be written i or j (the activity field should be the decision basis).

Following the freedom to define new mathematical notations, which was already stressed in [10], it is probable that new mathematical notations, which are not yet planned by existing infrastructure of notations may appear. Software makers should make this notation agility possible as a systematic change of notation may bring a considerable cognitive strength.

How many sets of notations per language and where should notations be gathered? This question is not much researched. The approaches of OpenMath or LaTeX described above has permeated MathML-content and follows the distributed practice of the semantic-web encoding of ontologies through RDF so it seems appropriate. The idea of each is that notations are where symbols are being defined or at least point to it; indeed OpenMath has a notion of content-dictionary which authors can create. However these approaches only live in a single encoding. A more general approach would be to set the expectation of a notations' table for each work of maths and that elements of this table are individually linkable so it is possible to augment the display of a formula with a link to the notation. Should such an approach become widespread, it will be much easier to discuss about notations and thus communicate about it, e.g. to take distance or to adopt.

7 Conclusion

In this paper we have illustrated the complex diversity of mathematical notations which is based on multiple cultures built by the traditions of mathematical knowledge texts. The diversity of the texts is rooted in the learning process of every reader who has learned the notations by learning the concepts around them and the allowed manipulations using them (their "haptics"). Because of this strong binding between learning and understanding the notations of a mathematical formula, we have argued that content specialists are needed to decide how to render mathematical formulæ in a way that is relevant to the readership.

Through multiple examples of diverging cultures, be them based on the language or on some other contextual information, we have shown the very rich set of notations and have argued that the set of relevant mathematical notations for a given software must be extensible and defined by experts close to content authors instead of employing general references such as [1]. As illustrated in the consideration of the history of mathematical knowledge such as [20,8], the evolution of mathematical notations has been an important fuel for the evolution of mathematical ideas. We have demonstrated that this invention potential is still there with even a creative industry aiming to explain mathematical formulæ better, for example in this systematization of mathematical augmentation [15].

Approaches to organize mathematical notations in interpretable bodies of knowledge are not yet widespread. The only initiatives we know of aim at rendering OpenMath or LaTeX [19,18,2]. It is likely that other collections of notations will appear in an effort to encode the diverse bodies of notations. This remains future work.

Among the current evolution of software handling mathematical notations, the development of recognition systems and datasets based on machine learning is growing. Such a system as [23] is working by grasping the context and is able to recognize sufficient information so that the symbols' interpretation, and probably more, is automatically detected. Doing so likely induces detecting the mathematical culture of the set of notations and concepts in the paper and in papers it depends upon. Expliciting the stacked effect of recognized cultural influences could have a similar expression than that of combining machine learning models by finetuning them on further annotation sets. Anchoring the culture ancestry goes along the lines of the suggestion of Timnit Gebru whom, when asked how to stimulate a more open participation to the body of science, suggested that each scientific contribution makes the effort of declaring all of the inspirations and background texts of the authors; this way induced biasses and untold assumptions could become easier to question [14].

Such a proposition goes in the same direction of tracking the many cultural influences so as to better understand an argument. This paper has demonstrated the need to make the decision of mathematical notations, and the power to invent new notations, in the hands of subject matter experts close to content authors and lave the door for newly invented notations.

And thus, science in the hands of the future generation can be seen to be promised to a bright future:

> *[...] enabling imagination to wander far beyond the tangible world we live in, and into the marvels of generality.*
> [20, chap 6, end of the 16^{th} paragraph].

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