# Groupes de présentation finie opérant dans une espace hyperbolique

(Finitely Presented Groups acting on a Hyperbolic Space)

A reading of Thomas Delzant's article, by Thomas Mattman & Paul Libbrecht

**Motivation/Goal:** extend the results of the earlier paper (Finitely Presented Subgroups of Hyperbolic Groups) to hyperbolic groups which are not necessarily torsion free. For example if $\Gamma$ is indecomposable as an amalgamated product or HNN extension over a finite group, we shall have:

**II.4.4 Lemma:** *Let $f : \Gamma \hookrightarrow \Gamma$ be an injective homomorphism. Then $\exists \gamma \in \Gamma$ of bounded length, $\psi \in Aut(\Gamma)$ and $\phi$ a "small" endomorphism of $\Gamma$ such that $f = C_\gamma \circ \phi \circ \psi$.*

The endomorhism $\phi$ is "small" in that the images of a previously given set of generators are bounded. Indeed, we can show,

**II.4.5 Theorem:** *$Aut(\Gamma)$ is finitely generated. It is generated by "small" automorphisms.*

**O.1.1 Definition of T(G)**

Let $P :< a_1, \ldots, a_n; R_1, \ldots, R_k >$ be a presentation of a group $G$. We define $T(P) = \sum_{1 \leq i \leq k} (|R_i| - 2)$ and $T = T(G)$ as the minimum over all presentations.

**O.1.2 Polyhedron:** A polyhedron consists of a graph and a set of faces. We call *embedded* faces, those faces attached by an injective map along their boundary.

**0.1.2 [sic!] The Relative Invariant**

Let $C$ be a family of subgroups of the group $G$. (Important example: $C$ is the set of subgroups of order less than a given integer $f$). We define $T(G, C)$ by requiring $T(G, C) \leq k$ if there is a simply connected polyhedron $P$ satisfying the following conditions:

c) G acts on P without inversion,

a) There is a finite number of faces in $P/G$ and the sum of their perimeters less 2 is $k$.

b) The vertex stabilizers of $P$ are conjugate to subgroups of elements of $C$.

**O.1.3** Such a $P$ (along with a maximal subtree in $P/G$) is called a *presentation of $G$ modulo $C$.*

**Examples:**

a) the universal cover of the van Kampen complex

b) Let $G = \Pi(\overline{X})$ where $\overline{X} = (X, (G_s)_{s \in X^0}, (C_y)_{y \in X^1}))$ is a graph of groups. Then the Serre tree is a presentation of $G$ modulo the vertex groups and $T(G, (G_s)_{s \in X^0}) = 0$.

**0.1.5** An edge $e \in P$ is *free* if $\overline{e}$ is not adjacent to a face of $P/G$. If, in addition, its stabilizer is the same as that of its two vertices, it is *trivial*

**O.1.6** An $f$-presentation of $G$ is a presentation modulo the subgroups of order less than $f$.

**O.1.7 Proposition:** *Let $G$ be finitely presented. Let $m(\tau; f)$ be the number of isomorphism classes of $f$-presentations of $G$ such that $P$ has no trivial edges nor digons, the number of faces modulo the $G$-action is less than $\tau$, and the faces are $k$-gons, with $3 \leq k \leq 9(3\tau - 1)$. Then $m(\tau; f)$ is finite.*

**Proof:** $P/G$ has at most $\tau$ faces, hence at most $9\tau(3\tau - 1)$ edges and $v = 18\tau(3\tau - 1)$ vertices. Let $\pi_\tau$ denote the number of such polyhedra. Then $m(\tau; f) \leq \pi_\tau^{\Gamma_f^v}$ where $\Gamma_f$ is the number of isomorphism classes of groups of order less than $f$. QED

### 0.2.1 Topological Operations Used

These operations take a presentation $P$ of $G$ modulo $C$ to another such. We perform these operations on $P/G$ or equivariantly on $P$.

1) *Subdivision*

2) *Fundamental Fold.* Given any vertex $s_0$ bounding two neighbour edges of the boundary of a face. Call the other ends $s_1$ and $s_1'$. We want to collapse the two edges along with a portion of the face in between. Two cases:
   A) The vertices $s_1$ and $s_1'$ are not in the same G-orbit: We require that the stabilizer of the vertex obtained by identifying $s_1$ and $s_1'$ be in $C$:
   $$<G_{s_1}, G_{s_1'}> \in C$$
   B) The vertices $s_1$ and $s_1'$ are in the same G-orbit: In this case, $\exists g \in G \ni gs_1 = s_1'$. Again, the stabilizer of the vertex obtained by identifying $s_1$ and $s_1'$ should be in $C$:
   $$<G_{s_1}, g> \in C.$$
   Important example: suppression of a digon.
3) *Composed Folds:* A subdivision of one or two edges followed by a fold.

If necessary, at the end of each fold we do the following trivial suppressions (note that they do not change the presentation induced by any maximal tree):

$S_1$) Removal of a flat vertex: A vertex of valence two is *flat* if its stabilizer is the same as the two edges incident on it.

$S_2$) Removal of a trivial vertex of valence one or two: A vertex of valence one (resp. two) is *trivial* if it's not adjacent to any face and its stabilizer is the same as that of the edge (one of the edges) incident on it.

The *multiplicity* of an edge, $e$, in a face $F$ of $P$ is the number of edges of $F$ having image $\overline{e}$ in $P/G$. Multiplicity is not changed by the topological operations.

**O.2.2 Lemma:** *Let $P$ be a triangular presentation of $G$ modulo $C$ with $T(P) = t$.*

*a) Let $P_1$ be a presentation obtained from $P$ by a topological operation. We can choose a maximal tree in $P_1/G$ such that the generators have length at most 3 in terms of those of $P/G$ and vice versa.*

*b) Let $Q$ be a presentation obtained from $P$ by a sequence of operations and suppose that $Q$ has no vertices of valence 1 or 2. Then the number of faces of $Q$ modulo $G$ does not exceed $t$ and they are $k$-gons with $3 \leq k \leq 9(3t-1)$.*

**Proof:** a) Each of the operations is related to a Whithead transformation on the generators.
b) The number of faces can only decrease (i.e. on suppression of a digon.) Their multiplicity is at most three so they are $k$ - gons with $k \leq 3e$, where $e$ is the number of non-free edges in $Q/G$. Let $r$ be the number of generators of $P$, excluding free edges. Since $Q/G$ is obtained from $P/G$ by a sequence of Whitehead moves, it also has at most $r$ non-free generators. So $e \leq v - 1 + r$, $v$ being the number of vertices in $P/G$. Since $Q/G$ has no vertices of valence one or two, $v \leq 2e/3$ and so $e/3 \leq r - 1$ and $k \leq 9(r-1)$. But $r$ is bounded by $3t$, the number of face edges in $P/G$. QED

**O.2.3** Consider a sequence $P_0, \ldots, P_n$ of $f$-presentations of $G$, $P_i$ obtained from $P_{i-1}$ by a fold. If $\tau = T(P_0)$, then O.1.7 shows that we may choose $n \leq m(\tau; f)$.

**Lemma:** *Let $A = \{a_1, \ldots, a_r\}$ be the generators of $P_0$ and $B = \{b_1, \ldots, b_s\}$ the generators of $P_n$. There is a $\psi \in Aut(G)$ with $|\psi(a_i)|_B \leq 3^{m(\tau;f)}$ and $|b_i|_{\psi(a_i)} \leq 3^{m(\tau;f)}$.*

# I. & II. Rips' Property & Acylindricity

**Def.** Let $G$ be finitely presented and fix a presentation $< a_1, \ldots, a_n | R_1, \ldots, R_T >$ . We will use the notation $a_0 = 1$ and $a_{-i} = a_i^{-1}$. Let $H$ be a simply-connected metric space (usually it will be hyperbolic, but this is not a requirement.) Let $\rho : G \to Isom(H)$ be an action of $G$ on $H$ by isometry.
We say $\rho$ is $(A, \alpha, \beta)$-*Rips* if there is an $h_0 \in H$ and, for each $-n \leq i \leq n$ an integer $L_i$ and a map $A_i : [0, L_i] \to H$ such that

1) $A_i(0) = h_0$,
2) $L_i = L_{-i}$ and $A_i(t) = \rho(a_i^{-1})A_i(L_i - t)$
3) If $R : a_i a_j a_k = 1$ is one of the relations, the triangle $A_i$, $\alpha_i A_j$, $\alpha_i \alpha_j A_k$ is $A$-*flat* in the sense that

$$\begin{cases} A_i(t) = A_{-k}(t) & \text{if } 0 \leq t \leq \frac{1}{2}(L_i + L_k - L_j - A) \\ A_{-i}(t) = A_j(t) & \text{if } 0 \leq t \leq \frac{1}{2}(L_i + L_j - L_k - A) \\ A_{-j}(t) = A_k(t) & \text{if } 0 \leq t \leq \frac{1}{2}(L_j + L_k - L_i - A) \end{cases}$$

4) The paths $A_i$ are $(\beta, \alpha)$-quasigeodesics, i.e. If $t$ and $u$ are integers, then:

$$\beta|t - u| \leq |A_i(t) - A_i(u)| \leq \alpha|t - u|$$

**I.2 Examples:**

**I.2.1 Trees:** If $H$ is a tree, every group action is $(0, 1, 1)$-Rips. Just take $h_0$ to be any point in $H$ and $A_i$ to be the segment $[h_0, a_i h_0]$ parametrized by arc length.

**I.2.2 Canonical Representatives:** Let $H$ be the Cayley 2-complex of a torsion-free $\delta$-hyperbolic group $\Gamma$ and $\rho : G \to \Gamma$ a homomorphism. Rips and Sela have shown $\rho$ is $(\varepsilon \cdot T, \varepsilon, v_{2\delta})$-Rips where $T$ is $T(G)$ $\varepsilon = 2^{v_{2\delta} 10^6 \delta^2} 10^7 \delta^2$ (a constant depending only on $\Gamma$) and $v_r$ is the number of elements in a ball of radius $r$.

**I.2.3 Rips Polyhedron:** Let $H = P_{20\delta}$ be the Rips' polyhedron of a hyperbolic group $\Gamma$. Since $20\delta \geq 4\delta + 1$, $P_{20\delta}$ is contractible hence simply connected. If $\rho$ is induced by a homomorphism $G \to \Gamma$ it is again $(\varepsilon \cdot T, \varepsilon, v_{2\delta})$-Rips. This comes out of their proof also, but applies for any hyperbolic group.

**Stupid examples:** Given an isometric action of $G$ on a one-connected complete metric space $H$, we can choose $h_0$ to be any point and $A_i$ to be any geodesic path from $h_0$ to $\rho(a_i) \cdot h_0$ (for positive $i$, which we then extend to negatives). Then we can choose $A$ to be big enough so that the flatness has no more sense works. We see thus that the interest is to bound $(A, \alpha, \beta)$ in some "independent" fashion.
Another stupid example would be the trivial action on any space, which is $(0, 0, 0)$-Rips.

**II.1 Acylindricity**

**II.1.1 Def.** We say $\rho$ is $\lambda$-acylindrical (resp. $\lambda$-$f$-acylindrical) if the stabilizer of a (bounded, non-empty) subset of diameter greater than or equal to $\lambda$ is trivial (resp. of order less than $f$).

**Examples:**

**II.1.2** If $H$ is a tree, this corresponds to Sela's definition (???).

**II.1.3** If $G$ has a decomposition as a graph of groups, with vertex stabilizers of order less than some integer $f$, then the action of $G$ on the Serre tree is $(1, f)$-acylindrical (??? we doubt ???).

No trivial action of an infinite group on an unbounded space is acylindrical.

**II.1.4** Let $G$ be finitely presented. Let $\Gamma$ be hyperbolic and $\rho : G \hookrightarrow \Gamma$ an injective homomorphism. Then the corresponding action on the $P_{10\delta}$ Rips' polyhedron is $(0, v_{10\delta})$-acylindrical. Indeed suppose $B$ is a bounded subset of $P_{10\delta}$. Then $\exists g \in P_{(0)} \ni d(g, B) \leq 10\delta$. Then, $\forall \sigma \in \Sigma = Stab(B)$, $d(\sigma(g), B) = d(\sigma(g), \sigma(B)) = d(g, B) \leq 10\delta$. So $\Sigma g$ is finite. But $G$ acts freely on vertices, so $\Sigma g$ is in bijection with $\Sigma$. But $\Sigma$, being finite, is then conjugate to a subgroup contained in the ball of radius $10\delta$.

**I.5.2 Induced Map:** In this paragraph, given a $G$-action on $H$ that satisfies the $(A, \alpha, \beta)$-Rips property relatively to a presentation $< a_1, \ldots, a_r \,|\, R_1, \ldots, R_T >$ we construct a map $A_\rho : \tilde{\Pi} \hookrightarrow H$; we mean $\tilde{\Pi}$ to be the universal cover of the van Kampen two-complex given by this presentation.

- the zero-skeleton of $\tilde{\Pi}$ is $G$ so we set $A_\rho(g) = \rho(g) \cdot h_0$. This is equivariant.
- the edges of $\tilde{\Pi}$ are pairs $(g, a_i) \in G \times \{a_1, \ldots, a_r\}$. Calling $\gamma_{g,i} : [0, 1] \longrightarrow \tilde{\Pi}$ its parametrization, we put $A_\rho(\gamma_{g,i}(t)) = \rho(g) \cdot A_i(t \cdot L_i)$. This is equivariant too.
- the faces $\Delta$ of $\tilde{\Pi}$ are triangles bounded by a sequence of three edges $(g, a_i)(g \cdot a_i, a_j)(g \cdot a_i \cdot a_j, a_k)$ which form a loop. The image under $A_\rho$ of this loop is then again a loop, but $H$ is simply connected, hence we have a map $\eta : \Delta \longrightarrow H$ which coincides with $A_\rho$ on the boundary; choosing a set of faces transversal to the action (i.e. such that there is exactly one in each orbit of faces), we may extend $A_\rho$ on all faces by choosing the $\eta$'s to be defined as translates of the $\eta$ given on the corresponding element of the transversal. Thus $A_\rho$ is equivariant.

It is important to note that we have a freedom of choice in defining $\eta$'s for each element of the transversal. Given a relator $a_i \cdot a_j \cdot a_k$ (or a cyclic permutation of it), the Rips property provides us with the following equality:

$$A_\rho(\gamma_{g,a_i}(t)) = \rho(g) \cdot A_i(t \cdot L_i) = \rho(g \cdot A_i \cdot a_j) \cdot A_k(L_k - L_i \cdot t)) = A_\rho(\gamma_{g \cdot a_i \cdot a_j, a_k}(1 - \frac{L_i}{L_k} \cdot t))$$

if $t \cdot L_i \in \mathbb{N} \cap [0, \frac{1}{2} \cdot (L_i + Lj - L_k - A)]$. We may thus DECIDE to map the points inside the triangle that are on the segment between these points to the same image $A_\rho(\gamma_{g,i}(t))$.

What we shall do now is to factor this equivariant map $A_\rho : \tilde{\Pi} \longrightarrow H$ through equivariant foldings of $\tilde{\Pi}$ that will make the action look more "simple".

**I.5.2 Construction of the "foliation" $\Lambda$**

Given an $(A, \alpha, \beta)$-Rips action $\rho : G \to Isom(H)$ we construct a graph $\Lambda$ in $\tilde{\Pi}$ which is invariant under the action of $G$ on $\tilde{\Pi}$ and which will be used to construct our graph of groups. The function $A_\rho : \tilde{\Pi} \to H$ will collapse each component of $\Lambda$ to a point. For a face $\Delta$ of $\tilde{\Pi}$ having boundary $(g, a_{i_1})$ ‘ $(ga_{i_1}, a_{i_2})$ ‘ $(ga_{i_1}a_{i_2}, a_{i_2})$, we define $\Delta \cap \Lambda$ as a set of *segments*. For each $k \in Z/3Z$ we have the segment $s \in [0, 1] \mapsto s \cdot t \cdot v_{i_k} + (1 - s) \cdot -t \cdot v_{i_{k-1}}$ provided $t \cdot L_{i_k} \in \mathbb{N}\frac{1}{2}(L_{i_k} + L_{i_{k-1}} - L_{i_{k-2}} - A)$.

**II.2 Rigidity Lemma** *Suppose $\rho : G \to Isom(H)$ is an $(A, \alpha, \beta)$-Rips, $(\lambda, f)$-acylindrical action, and $\Lambda_u$ is a connected component of $\Lambda$ in $\tilde{\Pi}$ such that for each edge $(g, a_i)$ intersecting $\Lambda_u$ in $g \cdot \gamma_i(l/L_i)$ with $l \leq \frac{1}{2} \cdot (L_i + L_j - L_k - A)$ we have $l \geq [\frac{\lambda}{2\beta} + 1]$ and $l \leq \frac{1}{2} \cdot (L_i + L_j - L_k - A) - [\frac{\lambda}{2\beta} + 1]$ (the square bracket stands for integer truncation). Then the stabilizer of any connected union $C$ of segments of $\Lambda_u$ is of order less than or equal to $f$.*

**Proof:** Let $(g, a_i)$ be an arc intersecting $\Lambda_u$ in $\rho(g) \cdot \gamma_i(l)$ with $l$ as given. We can choose $\Lambda_{u_0}, \ldots, \Lambda_{u_j}$, connected components of $\Lambda$, passing through $\gamma_i(l + m)$ where $m \in [l - [\frac{\lambda}{2 \cdot \beta} + 1], l + [\frac{\lambda}{2 \cdot \beta} + 1]]$. Now observe that all these components are "equivariantly parallel" in the sense that their images under $\pi$ are all isotopic. Thus we have a family $C_0$, …, $C_j$ of "equivariantly parallel" copies of $C$ on each component of $\Lambda$.
Finally the stablizer of $\cup C_k$ is of order $\leq f$ because its image under $A_\rho$ is a bounded set (it is contained in the image of a segment) and has diameter $\geq \lambda$. Indeed, choosing a point $c$ in $C$ and naming its "copies" $(c_k)_{k=1}^{j}$:

$$\text{diam } (A_\rho(\cup C_k)) \geq d_H(A_\rho(c_0), A_\rho(c_j)) \geq \beta \cdot ([l + \frac{\lambda}{2 \cdot \beta}] - [l - \frac{\lambda}{2 \cdot \beta}])$$

The image under $A_\rho$ of the union of these components together with one edge contains a segment of length

$\beta \cdot 2 \cdot (2 + [\frac{\lambda}{2\cdot\beta} + 1]) \geq 2 \cdot \beta + \beta \cdot 2 \cdot \frac{\lambda}{2\cdot\beta} = \lambda + 2 \cdot \beta \geq \lambda$ and is therefore of diameter greater than $\lambda$, hence the stabilizer is of order $\leq f$.

It remains to see that the stabilizer of this union is the same as the stablizer of $\Lambda_u$. This is clear because they are equivariantly parallel and the action of $G$ is by covering transformations, so that the stabilizer of the union is contained in the image of the fundamental group of $\pi(\cup C_i)$. But this subgroup is the same as that of $\pi(C)$. QED

**II.4 Main Theorem** *Let $G$ be a group equipped with a triangular presentation $< a_1, \ldots, a_r \,|\, R_1, \ldots R_T >$ and with an action $\rho : G \longrightarrow Isom(H)$ which is $(A, \alpha, \beta)$-Rips and $(\lambda, f)$-acylindrical.*
*There is a sequence $P_1, \ldots, P_N$ of $f$-presentations $P_0 = \tilde{\Pi}$, ..., $P_N =: Q$ where $\tilde{\Pi}$ is the universal cover of the van Kampen complex of the given presentation and $Q$ admits an equivariant map onto a $G$-tree $T$ such that the edge-stabilizers have order $\leq f$. Moreover, we have:*

– $\displaystyle\sum_{v \in V(X)} T(G_v, \{G_e \,|\, s(e) = v\}) \leq T.$

– *the polyhedron $P_i$ is obtained from $P_{i-1}$ by an fold (more precisely a composed fundamental fold followed possibly by a flat vertex suppression)*

– *for all vertices $v \in V(X)$ there is a point $p_v \in H$ and a set of generators $b_{v,1}, \ldots, b_{v,n_v}$ of $G_v$ such that:*

$$d_H(\rho(b_{v,i}) \cdot p_v,\ p_v) \leq 7 \cdot T \cdot \alpha \cdot (A + \frac{\lambda}{\beta} + 2)$$

– *if $M$ is such that $d_H(\rho(a_i) \cdot h_0,\ h_0) \leq M$ for all $i$, then we have $N \leq \frac{3\cdot M\cdot T}{2\cdot\beta}$.*

– *$\#V(X) \leq T + 1$ and* $\mathrm{val}(v) \leq 3 \cdot T(G_v, \{G_e \,|\, s(e) = v\}) \leq 3 \cdot T$ *for all vertices of $X$ except one.*

**Proof:** We construct the map $A_\rho$ and the graph $\Lambda$. The union of those components of $\Lambda$ that satisfy the hypothesis of the rigidity lemma will be called $\Lambda^*$. Let $F$ be the set of all faces of $\tilde{\Pi}$ that have $1 \in \tilde{\Pi} = G$ as one of their vertices. Note that we have exactly one angle (i.e. a pair composed of a face and a vertex of this face) of $\tilde{\Pi}/G$ for each angle $(1, \Delta)$ of $\tilde{\Pi}$ with $\Delta \in F$ and vice-versa.

For each angle $(1, \Delta)$

Choose the first segment $[s_1, s_1']$ of $\Lambda^*$ passing through this angle.

Subdivide the edges of this angle (i.e. the edges of the boundary of the face that touch the vertex), on the points $s_1$ and $s_1'$ and fold this angle.

Suppress all trivial vertices.

We recall that all our folds are done equivariantly, so that our algorithm has modified all angles which intersect a component of $\Lambda^*$. We have thus defined $P_i$ for $i = 1, \ldots, n$. Observe that we have a map $q_i : P_i \longrightarrow P_{i+1}$ given by the quotient map of the fold. It is equivariant and factors the map $A_\rho$ (i.e. there an $A_\rho^i : P_i \longrightarrow H$ such that $A_\rho = A_\rho^i \circ q_i \circ \ldots \circ q_1$). Call $p_i$ the composition $q_i \circ \ldots \circ q_1$. They are not simplicial but at least map vertices to vertices.

Note also that we have started with an $f$-presentation (even with a 1-presentation), and that at each step we have an $f$-presentation because for each vertex $v$ of a $P_i$, only one of two cases can happen: either $v$ is the image under $p_i$ of a union $K$ of segments of a component $\Lambda_u$ of $\Lambda^*$, or it is not. If it is then the stabilizer of $v$ in $P_i$ is the stabilizer of $K$ in $\tilde{\Pi}$, hence is a subgroup of order $\leq f$. If it is not, then then $p_i^{-1}(v)$ is a vertex of $\tilde{\Pi}$. Its stabilizer is then trivial.

We construct the tree: call two faces of $Q = P_N$ *neighbours* if there is a sequence of faces between those two such that two succesive elements of the sequence share an edge. This is clearly an equivalence relation. We call $T$ the result of collapsing each equivalence class of $Q$. $T$ is a graph. It is still connected. Moreover the equivalence classes are connected; hence $T$ is a simply connected graph, i.e. a tree. Observe that our neighbourhood relation is compatible with the group action, so that the action of $G$ dscends to the tree and the collapsing map $c : Q \longrightarrow T$ is equivariant. So $G$ acts on a tree. We call $X$ the graph $X/G$, it is finite since $Q/G$ is finite.

$$\begin{array}{ccccccccc} \tilde\Pi & \xrightarrow{q_1} & P_1 & \xrightarrow{q_2} & \cdot s & \xrightarrow{q_N} & Q & \xrightarrow{c} & T \\ \downarrow \pi & & \downarrow \pi_{P_1} & & \vdots & & \downarrow \pi_Q & & \downarrow \pi_T \\ \Pi & \xrightarrow{r_1} & P_1/G & \xrightarrow{r_2} & \cdot s & \xrightarrow{r_N} & Q/G & \xrightarrow{d} & X \end{array}$$

Along the folds, a maximal tree has been constructed in $P_i/G$, the one of $Q/G$(i.e. the intersection) wil be call $\tau$ and $\tilde\tau$ will be a maximal lift of $\tau$ which contains the vertex $p_N(1)$. For each vertex $v$ of $X$, we choose $\tilde v$ a vertex in $c(\tilde\tau) \subset T$ which lifts $v$. For each edge $e$ of $X$, we choose a lift $\tilde e$ in $T$ which intersects $c(\tilde\tau)$. We may thus set: for $v \in V(X)$, $G_v = \text{Stab }(\tilde v)$ and for $e \in E(X)$, $G_e = \text{Stab }(\tilde e)$. Classical Bass-Serre theory proves that the graph of groups $(X, (G_v)_{v\in V(X)}, (G_e)_{e\in E(X)})$ is indeed a decomposition of $G$.

Let us compute stabilizers of edges and vertices of $T$: an edge $e$ of $T$ can be one of two things: either $(c\circ p_N)^{-1}(e)$ is an edge, in which case its stabilizer is trivial (we did not modify this edge during the folds); or $(c\circ p_N)^{-1}(e)$ is not an edge which means that $(c\circ q_N)^{-1}(e)$ is a non-empty union of components of $\Lambda^*$ which all intersect one edge. The rigidity lemma then proves that its stabilizer is a subgroup of order $\leq f$.

A vertex $\tilde v$ of $T$ is the collapse of a neighbourhood class $Q_{\tilde v}$ of faces or the image of a vertex bounding no face. First observe that $Q$ is the union of $Q_{\tilde v}$'s pasted together by edges (of the tree); the Seifert-van Kampen theorem then proves that each $Q_{\tilde v}$ is simply connected (otherwise $Q$ is not simply connected). Furthermore $Q_{\tilde v}/G_{\tilde v}$ is isomorphic to $\pi_Q(Q_{\tilde v})$. Then each vertex of $Q_{\tilde v}$ is the image under $p_N$ of a vertex of $\tilde\Pi$ or a component of $\Lambda^*$ hence their stabilizers are trivial or stabilizers of a component of $\Lambda^*$; the latter are stabilizers of a free edges of $Q$ i.e. the stabilizer of an edge of $T$ that starts (or ends...) on $\tilde v$. We have thus just proved that $Q_{\tilde v}$ with the action of $G_{\tilde v}$ is a presentation of $G_{\tilde v}$ modulo the family $\{G_{\tilde e}\,|\, s(\tilde e) = \tilde v\}$. Also the number of faces $Q_{\tilde v}/G_{\tilde v}$ is the number of faces of $\pi_Q(Q_{\tilde v})$. Using our choice $\tilde v$ for each vertex $v$ of $X$ we have then:

$$\sum_{v\in V(X)} T(G_v, \{G_e\,|\, s(e) = v\}) \leq \#(\text{faces of } Q/G)$$

But observe that our folds do not raise the number of faces of $P_i$ so that $\#(\text{faces of } Q/G) \leq T$.

Finally we get the two other inequalities about the graph easily: having folded a whole component of $\Lambda^*$ the vertex which is on the center of the angle (the last one which we folded) is a free edge of valence two and its stabilizer is allways of order $\leq f$; so it is suppressed. Thus there is no free vertex of $Q/G$ except the base vertex; thus the number of vertices is less than the number of faces plus one i.e. $\leq T+1$. The second inequality follows from the fact that the edges going out of a neighbourhood component of $Q$ are edges created by a fold, which become free after the fold, unless the component in $Q/G$ contains some free loop (in which case it contains the image of the base-point under $p_N$) so that the number of edges going out is less than three times the number faces of this neighbourhood component of $Q/G$ (??).

Given a vertex $v \in V(X)$, we want to provide the set of generators for $G_v$ which will move points "reasonably". Observe that $\text{Stab}(Q_{\tilde v})$ is the stabilizer of $p_N^{-1}(Q_{\tilde v})$ which we call $\Pi_{\tilde v}$. Now this set is still simply connected hence is the universal cover of $\pi(\Pi_{\tilde v})$. If we subdivide $\tilde\Pi$ adding the innermost segments of $\Lambda^*$ for each triangle, then $\Pi_{\tilde v}$ is a subcomplex. Let us call $\Xi_{\tilde v}$ a lift of a maximal tree in the subcomplex $\pi(\Pi_{\tilde v})$ (with added segments). Then generators of $G_v$ are given by the set of elements of $G_v$ that realize the missing edges of the maximal tree in $\pi(\Pi_{\tilde v})$.

We realize these as desck transformations. Fix any vertex $p_v$ in $\Xi_{\tilde v}$. The images of $\Xi_{\tilde v}$ by elements of $G_v$ are in bijective correspondance with $G_v$; an image of $\Xi_{\tilde v}$ for which an edge in $\Pi_{\tilde v}$ exists with ends in both $\Xi_{\tilde v}$ and the image will be called surrounding; for each surrounding image we choose the element $b_{v,i}$ of $G_v$ that brings $\Xi_{\tilde v}$ to this image. This forms a set of generators as any element $g$ of $G_v$ gives rise to an edge path from $p_v$ to $g\cdot p_v$; the sequence of images of $\Xi_{\tilde v}$ that intersect this path provides the word to write $g$. The vertex $p_v$ is then sent by each generator $b_{v,i}$ inside a surrounding image of $\Xi_{\tilde v}$. We are thus left to bound the diameter of $A_\rho(U_{\tilde v})$ so that we shall have a bound for the images of $A_\rho(p_v)$ under the given generators of $G_v$. The images of the segments of $\Lambda^*$ by $A_\rho$ are points, thus only the edges which are in the boundary of a triangle count. As $\pi(\Xi_{\tilde v})$ is a tree, the number of those upstairs in $\Xi_{\tilde v}$ equals the number of those downstairs in $\pi(\Xi_{\tilde v})$, i.e. no more than $3\cdot T(G_v, \{G_e\,|\, s(e) = v\})$. The length of the image of one of these segments under $A_\rho$ is, by the Rips property, no more than $\alpha\cdot(A + \frac{\lambda}{\beta} + 2)$, hence the diameter of $A_\rho(\Xi_{\tilde v})$ is $\leq 3\cdot T(G_v, \{G_e\,|\, s(e) = v\})\cdot\alpha\cdot(A + \frac{\lambda}{\beta} + 2)$. Then the distance between $A_\rho(p_v)$ and $A_\rho(b_{v,i}\cdot p_v)$ is no more than twice the diameter of $A_\rho(\Xi_{\tilde{v}})$ plus the length of the "linking edge" which is $\leq \alpha \cdot (A + \frac{\lambda}{\beta} + 2)$ thus:

$$d_H(A_\rho(p_v),\ A_\rho(b_{v,i} \cdot p_v) \leq 6 \cdot (1 + T(G_v, \{G_e \mid s(e) = v\})) \cdot \alpha \cdot (A + \frac{\lambda}{\beta} + 2) \leq 6 \cdot (1 + T) \cdot \alpha \cdot (A + \frac{\lambda}{\beta} + 2)$$

But if $T(G_v, \{G_e \mid s(e) = v\}) = 0$ then $Q_{\tilde{v}}$ is a vertex. Hence this distance is zero; in all cases $7 \cdot T \cdot \alpha \cdot (A + \frac{\lambda}{\beta} + 2)$ is a bound for this distance.

Let us finally prove that the number of steps is bounded in terms of $M$, the maximum distance between $h_0$ and $a_i \cdot h_0$ in $H$. We need only count the number of segments of $\Lambda^*$ which we can do by counting the number of segments intersecting a face $\Delta$ of $Q$: having $d_H(\rho(a_i) \cdot h_0,\ h_0) \leq M$ implies $\beta \cdot L_i \leq M$ i.e. $L_i \leq \frac{M}{\beta}$. We count the number of segments crossing an "angle"; if the angle is one with edges $a_i$, $a_k$ and with $a_j$ as the opposite edge, then there are no more than $\frac{1}{2} \cdot (L_i + L_k - L_j - A) - \frac{\lambda}{\beta}$ crossing segments hence the total number of segments crossing this face is the sum over the three angles: $\frac{1}{2} \cdot (L_i + L_j + L_k) - \frac{3}{2} \cdot A - 3 \cdot \frac{\lambda}{\beta}$, so that the number of segments of $\Lambda^*$ is less than:

$$T \cdot \left( \frac{3}{2} \cdot \frac{M}{\beta} - \frac{3}{2} \cdot A - 3 \cdot \frac{\lambda}{\beta} \right) \leq \frac{3 \cdot M \cdot T}{2 \cdot \beta}$$

As we do exactly one composite fold per segment, this bound is a bound for $N$. QED

**II.4.4 Corollary** *Let $\Gamma$ be a $\delta$-hyperbolic group, $G$ indecomposable over finite groups and $f : G \longrightarrow \Gamma$ a monomorphism, let $P_{20\delta}$ be the Rips polyhedron of $\Gamma$. We know that the action of $G$ is $(\varepsilon \cdot T, \varepsilon, v_{2\delta})$-Rips and $(0, v_{10\delta})$ acylindrical, and let $M$ be the maximum of the $l(f(a_i))$. There is a vertex $\gamma \in P_{20\delta}$ and a new system of generators $\{b_1, \ldots, b_m\}$ such that $|\gamma^{-1} \cdot f(b_i) \cdot \gamma|_\Gamma = d(f(b_i) \cdot \gamma, \gamma) \leq 9 \cdot T \cdot \varepsilon \cdot A$.*
*But the $b_i$'s are obtained in terms of the $a_i$'s by at most $N = min \left\{ m(T, v_{10 \cdot \delta}, 3 \cdot T \cdot \frac{M}{v_{10\delta}} \right\}$ folds. So we have an automorphism $\psi$ such that the $|\psi(a_i)|_{b_i} \leq 3^N$ and $|\gamma^{-1} \cdot f \circ \psi(a_i) \cdot \gamma|_\Gamma \leq 3^N \cdot 9 \cdot T \cdot \varepsilon \cdot A$.*

*Thus is $G$ is $\Gamma$ we have just proved that the set of automorphisms of a torsion free hyperbolic group is finitely generated by the set of folds.*

**Concluding remarks**

More general results are deduced from this Main Theorem similarly to the ones we have seen during the course except we can remove the torsion free hypothesis and change every free product by an amalgamated product over a finite group. In particular, this solves the isomorphism problem for all hyperbolic groups.

The article that we study here does more: it has many applications to the computation of free decompositions; but the scope of our presentation was this theorem.

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